Fourier series

Every periodic continuous-time signal can be written as a sum of sinusoids

\[ x(t + T) = x(t) \]

Authe signals that have period \( T \)?

\[ \cos \]

\[ \cos \omega_0 t = \cos \frac{2\pi t}{T} \]

\[ \cos 2\omega_0 t = \cos \frac{4\pi t}{T} \]

\[ \rightarrow \cos k\omega_0 t \text{ } (k \text{ is int}) \]

\[ \rightarrow \text{ complex signal} \]

\[ e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t \]

\[ \sqrt{\text{he would do}} \]

\[ \sqrt{\text{similarly}} \]

\[ \text{are } e^{j\omega_0 t} \text{ is also periodic with period } T \]

\[ \text{but that charge amplitude, any complex number} \]

\[ x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\omega_0 t} \]

[Diagram of a signal and its Fourier series components]
Goal: represent \( x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} \)

--- Fourier series

How to find \( a_k \)'s? After integrating both sides:

\[ a_k = \frac{1}{T} \int_{T} x(t) e^{-jk\omega t} dt \]

\( a_k \) are the Fourier series coefficient of \( x(t) \)

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demo applet (Reznikov's website on Galadriel)

- square: more true, best fit
  - cutoff decreasing: normal, \( k \approx 25 \) responsible for \( 5x_{\text{vol}} \)
    - magnitude
      - we don't want very big
    - function of square is non-uniform (non-differentiable)
      - there is a shoulder

- triangle: looks like sine, complete
  - from \( a_k \) and it fits

- sawtooth: same shoulder because non-continuous
  - square shift button
  - magnitude doesn't change but phase does
  - same with cosine and sine

- Gibbs: we can't go below 95% of the height
Fourier transform

Discrete

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DFT samples the DTFT in the frequency domain.

\[ x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{j2\pi \frac{kt}{T}} + \]

DFP signal, \( X_k \) is a complex number.

\[ X_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi \frac{kt}{T}} dt \]

\[ X_k = \sum_{n=-\infty}^{\infty} x(n) \delta(t-nT) \]

\[ x(t) \text{ is a periodic signal with period } T \]

\[ x(t) \]

\[ x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{j2\pi \frac{kt}{T}} \]

if \( x(t) \) is a periodic function, we need an infinite number of \( N \) values.

let's try

\[ x(n) = \sum_{k=-\infty}^{\infty} x(k) e^{j2\pi \frac{k}{N} n} \]

\[ x(n) = \sum_{k=-\infty}^{\infty} x(k) e^{j2\pi \frac{k}{N} n} \quad n = 0, 1, \ldots, N-1 \]

doesn't work: discrete \( n \) and summing an infinite number of values?

work around

\[ e^{j2\pi \frac{k}{N} (n+N)} = e^{j2\pi \frac{k}{N} n} \quad \text{for } L = \text{int} \]

\[ e^{j2\pi \frac{k}{N} n} \]

\[ e^{j2\pi \frac{k}{N} n} = e^{j2\pi \frac{k}{N}} \quad \text{always} = 1 \]

Here are only \( N \) unique complex \( x(k) \) of period \( N \)

\[ x(n) = \sum_{k=-\infty}^{\infty} x(k) e^{-j2\pi \frac{k}{N} n} \quad k = 0, 1, \ldots, N-1 \]

\[ x(n) = \sum_{k=-\infty}^{\infty} x(k) e^{-j2\pi \frac{k}{N} n} \quad k = 0, 1, \ldots, N-1 \]
Let's rewrite

\[ W_N = e^{-j \frac{2\pi}{N}} \]

\[ (W_N)^N = 1 \]

\( W_N \) is the \( N \)-th root of 1

We rewrite

\[ x(l) = \sum_{n=0}^{N-1} x(n) W_n^l \]

We calculate

\[ W_1, W_2, W_3, W_4 \]

\[ W_1 = e^{2\pi i / 4} = e^{-\frac{\pi}{2}} \]

\[ W_2 = e^{-\pi i} \]

\[ W_3 = e^{-\frac{3\pi}{2} i} \]

\[ W_4 = e^{-2\pi i} \]

Radius \( r \) and \( \theta \)

\[ W_1 = \frac{e^{2\pi i / 4}}{4} = e^{-\frac{\pi}{2}} \]

\[ W_2 = e^{-\pi i} \]

\[ W_3 = e^{-\frac{3\pi}{2} i} \]

\[ W_4 = e^{-2\pi i} \]